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Bull. Sci. math. 132 (2008) 1–6

BULLETIN DES SCIENCES MATHÉMATIQUES

www.elsevier.com/locate/bulsci

Reduction of structure group of pullbacks of semistable principal bundles

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Received 2 January 2007; accepted 4 January 2007

Available online 21 February 2007

Abstract

Let C be an irreducible smooth projective curve defined over an algebraically closed field k. Let G be a semisimple linear algebraic group defined over the field k and $P \subset G$ a proper parabolic subgroup. Fix a strictly anti-dominant character χ of P. Let E_G be a semistable principal G-bundle over C. If the characteristic of k is positive, then E_G is assumed to be strongly semistable. Take any real number $\epsilon > 0$. Then there is an irreducible smooth projective curve $\widetilde{C}$ defined over k, a nonconstant morphism

$$\phi : \widetilde{C} \longrightarrow C,$$

and a reduction of structure group $\widehat{E}_P \subset \phi^* E_G$ of the principal G-bundle $\phi^* E_G$ to the subgroup P, such that the following holds:

$$\frac{\operatorname{degree}(\widehat{E}_P(\chi))}{\operatorname{degree}(\phi)} < \epsilon,$$

where $\widehat{E}_P(\chi)$ is the line bundle over $\widetilde{C}$ associated to the principal P-bundle $\widehat{E}_P$ for the character χ of P.

MSC: 14L15; 14H60

Keywords: Principal bundle; Semistable bundle; Parabolic group; Reduction

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0007-4497/$ – see front matter
doi:10.1016/j.bulsci.2007.01.001

1. Introduction

Let k be an algebraically closed field of arbitrary characteristic. Let G be a semisimple linear algebraic group defined over k. Fix a proper parabolic subgroup P of G. We also fix a strictly anti-dominant character χ of P.

Let E_G be a principal G-bundle on an irreducible smooth projective curve C defined over the field k. Take any triple (Y, ψ, E_P), where Y is an irreducible smooth projective curve defined over k,

$$\psi : Y \longrightarrow C$$

is a nonconstant morphism, and

$$E_P \subset \psi^* E_G$$

is a reduction of structure group of the principal G-bundle $\psi^* E_G$ over Y to the subgroup P of G. Let $E_P(\chi)$ be the line bundle over Y associated to the principal P-bundle E_P for the character χ of P.

Assume that for all triples (Y, ψ, E_P) of the above type (but for fixed P and χ) the inequality

$$\text{degree}\big(E_P(\chi)\big) \geqslant 0$$

holds. In [1] it was proved that such a principal G-bundle E_G is strongly semistable; see [1, p. 772, Proposition 4.3].

Our aim here is to prove the following theorem (see Theorem 2.3):

Theorem 1.1. *Fix P and χ as above. Let E_G be a semistable principal G-bundle over an irreducible smooth projective curve C defined over k. If the characteristic of k is positive, then E_G is assumed to be strongly semistable. Take any real number $\epsilon > 0$. Then there is an irreducible smooth projective curve $\widetilde{C}$ defined over k, a nonconstant morphism*

$$\phi : \widetilde{C} \longrightarrow C,$$

and a reduction of structure group $E_P \subset \phi^* E_G$ of the principal G-bundle $\phi^* E_G$ over $\widetilde{C}$ to the subgroup P, such that

$$\frac{\text{degree}(E_P(\chi))}{\text{degree}(\phi)} < \epsilon.$$

2. Reduction of structure group of a pullback

Let k be an algebraically closed field. Let C be an irreducible smooth projective curve defined over k. The genus of C will be denoted by g. If the characteristic of k is positive, then the Frobenius morphism $C \longrightarrow C$ will be denoted by F_C. For any integer $j \geqslant 1$, let

$$F_C^j := \overbrace{F_C \circ \cdots \circ F_C}^{j\text{-times}} : C \longrightarrow C \tag{1}$$

be the j-fold composition of the self-map F_C of C.

Let G be a semisimple linear algebraic group defined over the field k. Fix a proper parabolic subgroup P of G. Also, fix a strictly anti-dominant character χ of P. This means that the line bundle τ_χ over G/P defined by χ is ample. We recall that the sheaf of sections of the line bundle

τ_χ is the P-invariant direct image of the trivial line bundle over G on which P acts through the character χ.

Let E_G be a principal G-bundle over C. We recall that E_G is called *semistable* if for each proper parabolic subgroup $Q \subset G$, and each reduction of structure group $E_Q \subset E_G$ of the principal G-bundle E_G to the subgroup Q, the following holds: for each strictly anti-dominant character λ of Q, the line bundle over C associated to the principal Q-bundle E_Q for the character λ is of nonnegative degree. A principal G-bundle E_G over C is called *strongly semistable* if the pullback $(F_C^j)^* E_G$ is a semistable principal G-bundle for all j, where F_C^j is defined in (1). See [3,4] for more details.

For convenience, when the characteristic of k is zero, a semistable principal G-bundle over C will also be called a strongly semistable principal G-bundle.

Fix a strongly semistable principal G-bundle E_G over C. Consider the natural projection

$$f : E_G \longrightarrow E_G/P =: M. \tag{2}$$

It defines a principal P-bundle over the quotient space M. Let L be the line bundle over M associated to this principal P-bundle for the given character χ. We recall that the sheaf of sections of L is the P invariant direct image, for the projection f in (2), of the trivial line bundle over E_G on which the group P acts through the character χ.

Lemma 2.1. *Let E_G be a strongly semistable principal G-bundle over C. Let ξ be a line bundle over C such that* $\mathrm{degree}(\xi) > 2g$, *where* $g = \mathrm{genus}(C)$. *Then the line bundle $L \otimes f^*\xi$ on M is very ample, where f is the projection in* (2), *and L is defined above.*

Proof. Consider the G-module $V_\chi := H^0(G/P, \tau_\chi)$, where τ_χ, as before, is the line bundle over G/P defined by χ; the action of G on τ_χ, which is a lift of the left-translation action of G on G/P, induces an action of G on V_χ. The direct image f_*L over C is canonically identified with the vector associated to the principal G-bundle E_G for this G-module V_χ. The group G being semisimple does not admit any nontrivial character. In particular, $\bigwedge^{\mathrm{top}} V_\chi$ is a trivial G-module. Hence the line bundle

$$\det f_*L := \bigwedge^{\mathrm{top}} f_*L,$$

which is the line bundle over C associated to the principal G-bundle E_G for the trivial G-module $\bigwedge^{\mathrm{top}} V_\chi$, is trivializable. From the fact that E_G is strongly semistable it follows that the associated vector bundle f_*L over C is semistable [4, p. 288, Theorem 3.23].

Any ample line bundle over G/P is very ample. Therefore, from the given condition that the character χ of P is strictly anti-dominant it follows that the line bundle τ_χ over G/P defined by it is very ample. Hence the restriction of the line bundle $L \otimes f^*\xi$ to any fiber of the map f is very ample.

Let K_C denote the canonical line bundle of the smooth curve C. Fix k-rational points $x, y \in C$ which are not necessarily distinct. The vector bundle $\xi^* \otimes \mathcal{O}_C(x+y) \otimes K_C \otimes (f_*L)^*$ over C is semistable because f_*L is so. As $\mathrm{degree}(f_*L) = 0$ (recall that the line bundle $\det f_*L$ is trivializable), the given condition that $\mathrm{degree}(\xi) > 2g$ implies that

$$\mathrm{degree}\big(\xi^* \otimes \mathcal{O}_C(x+y) \otimes K_C \otimes (f_*L)^*\big) < 0.$$

Since $\xi^* \otimes \mathcal{O}_C(x+y) \otimes K_C \otimes (f_*L)^*$ is a semistable vector bundle of negative degree, we have

$$H^0\big(C, \xi^* \otimes \mathcal{O}_C(x+y) \otimes K_C \otimes (f_*L)^*\big) = 0.$$

Now the Serre duality gives

$$H^1\big(C, \xi \otimes \mathcal{O}_C(-x-y) \otimes f_*L\big) = 0. \tag{3}$$

Consider the short exact sequence of coherent sheaves

$$0 \longrightarrow \xi \otimes \mathcal{O}_C(-x-y) \otimes f_*L \longrightarrow \xi \otimes f_*L \longrightarrow (\xi \otimes f_*L)|_{x+y} \longrightarrow 0$$

on C. Let

$$H^0(C, \xi \otimes f_*L) \xrightarrow{\sigma} (\xi \otimes f_*L)|_{x+y} \longrightarrow H^1\big(C, \xi \otimes \mathcal{O}_C(-x-y) \otimes f_*L\big) \tag{4}$$

be the corresponding long exact sequence of cohomologies. Now, from (3) we know that the restriction homomorphism σ in (4) is surjective.

Therefore, the restriction homomorphism

$$H^0(C, \xi \otimes f_*L) \longrightarrow (\xi \otimes f_*L)_x$$

is surjective for each k-rational point x of C.

Also, using the projection formula it follows that

$$H^0(C, \xi \otimes f_*L) = H^0\big(M, (f^*\xi) \otimes L\big).$$

Combining these we conclude the following. The sections of the line bundle $L \otimes f^*\xi$ over M separate any pair of points of M, and also the sections separate tangent vectors.

In other words, the line bundle $L \otimes f^*\xi$ is very ample. This completes the proof of the lemma. □

Let E_G be a strongly semistable principal G-bundle over C. Fix a line bundle ξ over C with $\text{degree}(\xi) > 2g$.

Take any positive integer n. Since the character χ of P is strictly anti-dominant, it follows immediately that the character χ^n of P is also strictly anti-dominant. Therefore, replacing the character χ by the character χ^n in Lemma 2.1 we conclude that the line bundle $L^{\otimes n} \otimes f^*\xi$ over $M = E_G/P$ is very ample.

Let

$$Y_n \subset M \tag{5}$$

be a reduced smooth complete intersection curve obtained by intersecting hyperplanes in M lying in the very ample complete linear system $|L^{\otimes n} \otimes f^*\xi|$.

For a line bundle η on a smooth projective variety Z, by $[\eta]$ we will denote the image of η in the group defined by the divisors on Z modulo rational equivalence. If $\dim Z = \delta$, and $L_1, \dots, L_\delta$ are line bundles over Z, then

$$[L_1] \dots [L_\delta] \in \mathbb{Z}$$

will denote the degree of the intersection of divisor classes associated to $L_1, \dots, L_\delta$.

Proposition 2.2. *The degree of the restriction of the line bundle L to the curve Y_n (defined in (5)) is*

$$\text{degree}(L|_{Y_n}) = dn^{d-1} \cdot \text{degree}(\tau_\chi) \cdot \text{degree}(\xi),$$

where $d := \dim G/P$ and

$$\text{degree}(\tau_\chi) := [\tau_\chi]^d \in \mathbb{Z} \tag{6}$$

with τ_χ being the line bundle over G/P defined by the character χ.

Proof. Since $[L^{\otimes n} \otimes f^*\xi] = n \cdot [L] + f^*[\xi]$, and $\dim E_G/P = d+1$, we have

$$\mathrm{degree}(L|_{Y_n}) = [L]\big(n \cdot [L] + f^*[\xi]\big)^d \in \mathbb{Z},$$

where f is the projection in (2). As $[\xi]^2 = 0$, we have

$$[L]\big(n \cdot [L] + f^*[\xi]\big)^d = n^d \cdot [L]^{d+1} + dn^{d-1} \cdot \mathrm{degree}(\tau_\chi) \cdot \mathrm{degree}(\xi) \in \mathbb{Z}.$$

Hence we have

$$\mathrm{degree}(L|_{Y_n}) = n^d \cdot [L]^{d+1} + dn^{d-1} \cdot \mathrm{degree}(\tau_\chi) \cdot \mathrm{degree}(\xi). \tag{7}$$

On the other hand, we have

$$0 = [L]^{d+1} \in \mathbb{Z}$$

(see [2, p. 26, (6.3)]). Therefore, from (7) it follows that

$$\mathrm{degree}(L|_{Y_n}) = dn^{d-1} \cdot \mathrm{degree}(\tau_\chi) \cdot \mathrm{degree}(\xi).$$

This completes the proof of the proposition. □

Theorem 2.3. *Let E_G be a strongly semistable principal G-bundle over C. Fix a proper parabolic subgroup $P \subset G$, and also fix a strictly anti-dominant character χ of P. Take any real number $\epsilon > 0$. Then there is an irreducible smooth projective curve $\widetilde{C}$ defined over k, a nonconstant morphism*

$$\phi : \widetilde{C} \longrightarrow C,$$

and a reduction of structure group

$$\widehat{E}_P \subset \phi^* E_G$$

of the principal G-bundle $\phi^* E_G$ to the subgroup P, such that the following holds:

$$\frac{\mathrm{degree}(\widehat{E}_P(\chi))}{\mathrm{degree}(\phi)} < \epsilon, \tag{8}$$

where $\widehat{E}_P(\chi)$ is the line bundle over $\widetilde{C}$ associated to the principal P-bundle $\widehat{E}_P$ for the character χ of P, and $\mathrm{degree}(\phi) \in \mathbb{N}^+$ *is the degree of the map ϕ.*

Proof. Consider the complete intersection curve

$$\iota : Y_n \hookrightarrow E_G/P \tag{9}$$

in Proposition 2.2. Let

$$f_n : Y_n \longrightarrow C \tag{10}$$

be the restriction of the map f in (2) to Y_n. The above inclusion map ι gives a reduction of structure group

$$E_P^{(n)} \subset f_n^* E_G \tag{11}$$

of the principal G-bundle $f_n^* E_G$ over Y_n to the parabolic subgroup $P \subset G$. To see this reduction of structure group to P, consider the natural map

$$(f_n^* E_G)/P = f_n^*(E_G/P) \longrightarrow E_G/P.$$

Let

$$f_n^* E_G \longrightarrow (f_n^* E_G)/P \longrightarrow E_G/P$$

be the composition of it with the quotient map $f_n^* E_G \longrightarrow (f_n^* E_G)/P$. The subvariety $E_P^{(n)}$ in (11) is the inverse image of $Y_n \subset E_G/P$ for this composition map.

Let $E_P^{(n)}(\chi)$ denote the line bundle over Y_n associated to the principal P-bundle $E_P^{(n)}$ for the character χ of P. This line bundle $E_P^{(n)}(\chi)$ is evidently canonically identified with the pullback $\iota^* L$, where ι is the inclusion map in (9). Therefore, Proposition 2.2 gives that

$$\operatorname{degree}\bigl(E_P^{(n)}(\chi)\bigr) = dn^{d-1} \cdot \operatorname{degree}(\tau_\chi) \cdot \operatorname{degree}(\xi). \tag{12}$$

Now we will compute the degree of the map f_n in (10).

Take a general k-rational point $x \in C$. The degree of f_n is the cardinality of the intersection $f^{-1}(x) \cap D_1 \cap \cdots \cap D_d$, where $D_i \in |L^{\otimes n} \otimes f^* \xi|$ with $Y_n = D_1 \cap \cdots \cap D_d$; as before, $d = \dim G/P$ and f is the projection in (2). From this and the fact that $(f^*[\xi]) \cdot (f^*[\xi]) = 0$ we have

$$\operatorname{degree}(f_n) = f^*\bigl[\mathcal{O}_C(x)\bigr]\bigl(n \cdot [L] + f^*[\xi]\bigr)^d = n^d \cdot \operatorname{degree}(\tau_\chi), \tag{13}$$

where $\operatorname{degree}(\tau_\chi)$ is defined in (6).

Combining (12) and (13),

$$\frac{\operatorname{degree}(E_P^{(n)}(\chi))}{\operatorname{degree}(f_n)} = \frac{d \cdot \operatorname{degree}(\xi)}{n}.$$

Now we may take the pair $(\phi \widehat{E}_P)$ in the statement of the theorem to be the pair $(f_n, E_P^{(n)})$, where n is sufficiently large, to conclude that (8) holds. This completes the proof of the theorem. □

References

[1] I. Biswas, A.J. Parameswaran, A criterion for the strongly semistable principal bundles over a curve in positive characteristic, Bull. Math. Sci. 128 (2004) 761–773.

[2] I. Biswas, A.J. Parameswaran, S. Subramanian, Monodromy group for a strongly semistable principal bundle over a curve, Duke Math. J. 132 (2006) 1–48.

[3] A. Ramanathan, Stable principal bundles on a compact Riemann surface, Math. Ann. 213 (1975) 129–152.

[4] S. Ramanan, A. Ramanathan, Some remarks on the instability flag, Tôhoku Math. J. 36 (1984) 269–291.